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Weak and Strong Convergence for Common Fixed Points of a Finite Family of Asymptotically Nonexpansive Nonself-mappings

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ABSTRACT

In this paper, we established some weak and strong convergence theorems for a multi-step iterative scheme with errors for a finite families of asymptotically nonexpansive nonself-mappings in Banach spaces. Our results extended and improve the recent ones announced by Wang [Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, J. Math. Anal. Appl., 323(2006)550-557.] and Chidume, Ali [Approximation of common fixed points for finite families of nonself asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., 326(2007) 960-973.]

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INTRODUCTION

Let K be a nonempty subset of a real normed space E. A self mapping $T : K \to K$ is called *nonexpansive* if $\| Tx - Ty \| \le \| x - y \|$ for all $x, y \in K$ and *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for all $n \in \mathbb{N}$,

$$\| T^n x - T^n y \| \le k_n \| x - y \| \quad \text{for all } x, y \in K.$$

T is called *uniformly L-Lipschitzian* if there exists a real number $L > 0$ such that

$$\| T^n x - T^n y \| \le L\| x - y \| \quad \text{for all } x, y \in K.$$

and integers $n \ge 1$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk (1972) and the class forms an important genealization of that of nonexpansive mappings. It was prove by Goebel and Kirk (1972) that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self mapping on K, then T has a fixed point.

Iterative methods for approximating fixed points of nonexpansive mappings have been studied by many authors. In most of these papers, the well-known Mann iteration process (Mann, 1953), $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n \ , \quad n \ge 1, \tag{1}$$

has been studied and the operator T has been assumed to map K into itself. The convexity of K then ensures that the sequence $\{x_n\}$ generated by (1) is well defined. If, however, K is a *proper* subset of the real Banach space E and T maps K into E (as is the case in many applications), then the sequence given by (1) may not be well defined. One method that has been used to overcome this in the case of single operator T is to introduce a retraction $P : E \to K$ in the recursion formula (1) as follows: $x_1 \in K$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n PTx_n \ , \quad n \geq 1.$$

Recent results on approximation of fixed points of nonexpansive and asymptotically nonexpansive self and nonself single mappings can be found.

The concept of nonself asymptotically nonexpansive mapping was introduced by Chidume et al. (2003) as an important generalization of asymptotically nonexpansive self mappings.

Definition 1. (Chidume, Ofoedu, and Zegeye, 2003) Let K be a nonempty subset of a real normed space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \in \mathbb{N}$,

$$\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq k_n \| x - y \| \qquad \text{for every } x, y \in K.$$

T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq L \| x - y \| \qquad \text{for every } x, y \in K.$$

Using an Ishikawa-like scheme (Ishikawa, 1974), Takahashi and Tamura (1998) proved strong and weak convergence of a sequence defined by

$$x_{n+1} = \alpha_n S[\beta_n Tx_n + (1-\beta_n)\, x_n] + (1-\alpha_n)\, x_n$$

to common fixed point of a pair of nonexpansive mappings T and S. Recently, Wang (2006) was introduced and iteration scheme for approximating common fixed points of two nonself asymptotically nonexpansive mappings and to proved some strong and weak convergence theorem for such mappings in uniformly convex Banach spaces.

It is our purpose in this paper to introduce a mult-step iteration process with errors for approximating common fixed points for finite families of nonself asymptotically nonexpansive mappings. For these families of operators, we prove strong convergence theorem in uniformly convex Banach spaces, and prove weak convergence theorem in real uniformly convex Banach spaces that satisfy Opial's condition, or have Frechet differentiable norm. Our theorems generalize many recent results.

PRELIMINARIES

Let E be a real Banach space and J denote the normalized duality mapping from E to 2^{E^*} define by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\| \},$$

where E^* denotes the dual space of E and $\langle \cdot,\cdot \rangle$ denotes the generalized duality pairing between elements of E and E^*. Let E be a real normed linear space. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1,\ \varepsilon = \|x-y\| \right\}.$$

Notation. We use $\rightarrow$ for strong convergence and $\xrightarrow{w}$ for weak convergence.

E is called *uniformly convex* if and only if $\delta_E(\varepsilon) > 0,\ \forall \varepsilon \in (0,2]$. The norm of E is said to be Frechet differentiable if for each $x \in E$ with $\|x\| = 1$ the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly for y, with $\| y \| = 1$. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x\ \ \forall x \in K$. Every closed convex set of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y\ \ \forall y \in R(P)$, the range of P. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \xrightarrow{w} x^* \in D(T)$ and $Tx_n \to p$ then $Tx^* = p$.

A mapping $T : K \to K$ is said to be *semicompact* if, for any bounded sequence $\{x_n\}$ in K such that $\| x_n - Tx_n \| \to 0$ as $n \to \infty$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some x^* in K. T is said to be *completely continuous* if for every bounded sequence $\{x_n\}$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that the sequence $\{Tx_{n_j}\}$ converges to some element of the range of T.

A Banach space E is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \xrightarrow{w} x$ implies that

$$\liminf_{n \to \infty} \|x_n + x\| < \liminf_{n \to \infty} \|x_n + y\| \qquad \forall y \in E,\ \ y \neq x.$$

In what follows we shall use the following results.

Lemma 2. (Tan and Xu, 1993) Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers such that $\lambda_{n+1} \le \lambda_n + \sigma_n\ \forall n \geq 1$, and $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \lambda_n$ exists.

Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$ then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 3. (Schu, 1991) Let E be a real uniformly convex Banach space and $0 \le p \le t_n \le q < 1$ for all positive integers $n \ge 1$. Suppose that x_n and y_n are two sequences of E such that

$$\limsup_{n\to\infty}\|x_n\| \le r, \quad \limsup_{n\to\infty}\|y_n\| \le r \quad \text{and} \quad \lim_{n\to\infty}\|t_nx_n+(1-t_n)y_n\| = r$$

hold for some $r \ge 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 4. (Chidume *et al.*, 2003) Let E be a real uniformly convex Banach space, K a nonempty closed subset of E, and let $T : K \to E$ be asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$, then $(I - T)$ is demiclosed at zero.

Lemma 5. (Tan and Xu, 1993) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \le (1 + \delta_n)\, a_n + b_n, \qquad \forall n = 1,2,\ldots$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 6. (Kaczor, 2002) Let E be a real uniformly convex Banach space whose dual E^* satisfies the Kadec-Klee property Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in \omega_w(\{x_n\})$ (where $\omega_w(\{x_n\})$ denote the weak limit set of $\{x_n\}$).
Suppose $\lim_{n\to\infty}\|tx_n + (1 - t)x^* - y^*\|$ exists for all $t \in [0, 1]$. Then, $x^* = y^*$.

Lemma 7. (Falset *et al.*, 2001) Let E be a uniformly convex Banach space and K a convex subset of E. Then there exists a strictly increasing continuous convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for each Lipschitz mapping $S : K \to K$ with Lipschitz constant L, we have,

$$\|\alpha Sx + (1-\alpha)Sy - S(\alpha x + (1-\alpha)y)\| \le L\phi^{-1}\left(\|x-y\| - \frac{1}{L}\|Sx - Sy\|\right)$$

for all $x,y \in K$ and $0 < \alpha < 1$.

MAIN RESULTS

In this section we state and prove the main results of this paper. In the sequel, we designate the set 1, 2,..., N by I and we always assume $\bigcap_{i=1}^{N} F(T_i) \ne \phi$.
We now introduce the following iteration scheme:

$$
\begin{aligned}
&x_1 \in K,\\
&x_n^{(1)} = P((1-\alpha_n^{(1)})x_n + \alpha_n^{(1)} T_1 (PT_1)^{n-1} x_n + v_n^{(1)}),\\
&x_n^{(2)} = P((1-\alpha_n^{(2)})x_n + \alpha_n^{(2)} T_2 (PT_2)^{n-1} x_n^{(1)} + v_n^{(2)}), \qquad (2)\\
&\vdots\\
&x_{n+1} = x_n^{(N)} = P((1-\alpha_n^{(N)})x_n + \alpha_n^{(N)} T_N (PT_N)^{n-1} x_n^{N-1} + v_n^{(N)}), \quad n \ge 1.
\end{aligned}
$$

Lemma 8. Let E be a real uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with a retaction P. Let $T_1, T_2, ..., T_N : K \to E$ be asymptotically nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \ne \phi$ and $\left\{k_n^{(i)}\right\}_{n=1}^{\infty}$ satisfy $\sum_{n=1}^{\infty}(k_n^{(i)}-1) < \infty$ for all $i \in \{1,2,\ldots,N\}$. Let $\left\{\alpha_n^{(i)}\right\}_{n=1}^{\infty}$ be a sequence in $[\varepsilon, 1-\varepsilon], \varepsilon \in (0,1)$ for all $i \in \{1,2,\ldots,N\}$. Let $\{x_n\}$ be a sequence defined iteratively by (2). Then $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in \bigcap_{i=1}^{N} F(T_i)$.

Proof. Setting $k_n^{(i)} = 1 + \lambda_n^{(i)}, \forall i = 1,2,\ldots,N.$ Since $\sum_{n=1}^{\infty}(k_n^{(i)}-1) < \infty$, so $\sum_{n=1}^{\infty}(\lambda_n^{(i)}) < \infty, \forall i = 1,2,\ldots,N.$ For any $x^* \in \bigcap_{i=1}^{N} F(T_i)$, by (3.1), we have

$$
\begin{aligned}
\| x_n^{(1)} - x^* \| &= \| P((1-\alpha_n^{(1)})x_n + \alpha_n^{(1)} T_1 (PT_1)^{n-1} x_n + v_n^{(1)}) - x^* \|\\
&\le \|(1-\alpha_n^{(1)})(x_n - x^*) + \alpha_n^{(1)}(T_1(PT_1)^{n-1}x_n - x^*) + v_n^{(1)}\|\\
&\le (1-\alpha_{\mathrm{n}}^{(1)})\|x_n - x^*\| + \alpha_{\mathrm{n}}^{(1)}(1+\lambda_n^{(1)})\|x_n - x^*\| + \|v_n^{(1)}\|\\
&\le (1+\lambda_n^{(1)})\|x_n - x^*\| + \|v_n^{(1)}\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\| x_n^{(2)} - x^* \| &= \| P((1-\alpha_n^{(2)})x_n + \alpha_n^{(2)} T_1 (PT_1)^{n-1} x_n^{(1)} + v_n^{(2)}) - x^* \|\\
&\le (1-\alpha_n^{(2)})\|x_n - x^*\| + \alpha_n^{(2)} \|(T_1(PT_1)^{n-1}x_n^{(1)} - x^*\| + \|v_n^{(2)}\|\\
&\le (1-\alpha_n^{(2)})\|x_n - x^*\| + \alpha_n^{(2)}(1+\lambda_n^{(2)})\|x_n^{(1)} - x^*\| + \|v_n^{(2)}\|\\
&\le (1-\alpha_n^{(2)})\|x_n - x^*\| + \alpha_n^{(2)}(1+\lambda_n^{(2)})\left\{(1+\lambda_n^{(1)})\|x_n - x^*\| + \|v_n^{(1)}\|\right\}\\
&\quad + \|v_n^{(2)}\|\\
&\le \left\{1+\lambda_n^{(2)} + \lambda_n^{(1)}(1+\lambda_n^{(2)})\right\}\|x_n - x^*\| + (1+\lambda_n^{(2)})\|v_n^{(1)}\| + \|v_n^{(2)}\|.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\| x_n^{(3)} - x^* \| &\le (1-\alpha_n^{(3)})\|x_n - x^*\| + \alpha_n^{(3)}(1+\lambda_n^{(3)})\| x_n^{(2)} - x^* \| + \| v_n^{(3)} \| \\
&\le (1-\alpha_n^{(3)})\|x_n - x^*\| + \alpha_n^{(3)}(1+\lambda_n^{(3)})\{[1+\lambda_n^{(2)} + \lambda_n^{(1)}(1+\lambda_n^{(2)})]\|x_n - x^*\| \\
&\quad + (1+\lambda_n^{(2)})\| v_n^{(1)} \| + \| v_n^{(2)} \|\} + \| v_n^{(3)} \| \\
&= \left\{(1-\alpha_n^{(3)}) + \alpha_n^{(3)}(1+\lambda_n^{(3)})\left[1+\lambda_n^{(2)} + \lambda_n^{(1)}(1+\lambda_n^{(2)})\right]\right\}\|x_n - x^*\| \\
&\quad + \alpha_n^{(3)}(1+\lambda_n^{(3)})(1+\lambda_n^{(2)})\| v_n^{(1)} \| + \alpha_n^{(3)}(1+\lambda_n^{(3)})\| v_n^{(2)} \| + \| v_n^{(3)} \| \\
&\le \left\{1+\lambda_n^{(3)} + \lambda_n^{(2)}(1+\lambda_n^{(3)}) + \lambda_n^{(1)}(1+\lambda_n^{(3)})(1+\lambda_n^{(2)})\right\}\|x_n - x^*\| \\
&\quad + (1+\lambda_n^{(3)})(1+\lambda_n^{(2)})\| v_n^{(1)} \| + (1+\lambda_n^{(3)})\| v_n^{(2)} \| + \| v_n^{(3)} \|
\end{aligned}
$$

and

$$
\begin{aligned}
\| x_n^{(4)} - x^* \| &\le (1-\alpha_n^{(4)})\|x_n - x^*\| + \alpha_n^{(4)}(1+\lambda_n^{(4)})\| x_n^{(3)} - x^* \| + \| v_n^{(4)} \| \\
&\le (1-\alpha_n^{(4)})\|x_n - x^*\| + \alpha_n^{(4)}(1+\lambda_n^{(4)})\{[1+\lambda_n^{(3)} + \lambda_n^{(2)}(1+\lambda_n^{(3)}) \\
&\quad + \lambda_n^{(1)}(1+\lambda_n^{(3)})(1+\lambda_n^{(2)})]\|x_n - x^*\| + (1+\lambda_n^{(3)})(1+\lambda_n^{(2)})\| v_n^{(1)} \| \\
&\quad + (1+\lambda_n^{(3)})\| v_n^{(2)} \| + \| v_n^{(3)} \|\} + \| v_n^{(4)} \| \\
&= \{(1-\alpha_n^{(4)}) + \alpha_n^{(4)}(1+\lambda_n^{(4)})[1+\lambda_n^{(3)} + \lambda_n^{(2)}(1+\lambda_n^{(3)}) \\
&\quad + \lambda_n^{(1)}(1+\lambda_n^{(3)})(1+\lambda_n^{(2)})]\}\|x_n - x^*\| \\
&\quad + \alpha_n^{(4)}(1+\lambda_n^{(4)})(1+\lambda_n^{(3)})(1+\lambda_n^{(2)})\| v_n^{(1)} \| \\
&\quad + \alpha_n^{(4)}(1+\lambda_n^{(4)})(1+\lambda_n^{(3)})\| v_n^{(2)} \| + \alpha_n^{(4)}(1+\lambda_n^{(4)})\| v_n^{(3)} \| + \| v_n^{(4)} \| \\
&\le \{1+\lambda_{\mathrm{n}}^{(4)} + \lambda_{\mathrm{n}}^{(3)}(1+\lambda_n^{(4)}) + \lambda_n^{(2)}(1+\lambda_n^{(4)})(1+\lambda_n^{(3)}) \\
&\quad + \lambda_n^{(1)}(1+\lambda_n^{(4)})(1+\lambda_n^{(3)})(1+\lambda_n^{(2)})\}\| x_n - x^* \| \\
&\quad + (1+\lambda_{\mathrm{n}}^{(4)})(1+\lambda_{\mathrm{n}}^{(3)})(1+\lambda_{\mathrm{n}}^{(2)})\|v_n^{(1)}\| + (1+\lambda_{\mathrm{n}}^{(4)})(1+\lambda_{\mathrm{n}}^{(3)})\|v_n^{(2)}\| \\
&\quad + (1+\lambda_{\mathrm{n}}^{(4)})\|v_n^{(3)}\| + \|v_n^{(4)}\|.
\end{aligned}
$$

By continuing the above method, we obtain

$$
\begin{aligned}
\| x_{n+1} - x^* \| &\le (1-\alpha_n^{(N)})\| x_n - x^* \| + \alpha_n^{(N)}(1+\lambda_n^{(N)})\| x_n^{(N\text{-}1)} \text{-} x^* \| + \| v_n^{(N)} \| \\
&\le \{1+\lambda_n^{(N)} + \lambda_n^{(N-1)}(1+\lambda_n^{(N)}) + \lambda_n^{(N-2)}(1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)}) + \dots \\
&\quad + \lambda_n^{(1)}(1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)})\dots(1+\lambda_n^{(2)})\}\| x_n - x^* \| \\
&\quad + (1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)})\dots(1+\lambda_n^{(2)})\| v_n^{(1)} \| \\
&\quad + (1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)})\dots(1+\lambda_n^{(3)})\| v_n^{(2)} \| + \dots \\
&\quad + (1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)})\| v_n^{(N-2)} \| + (1+\lambda_n^{(N)})\| v_n^{(N-1)} \| + \| v_n^{(N)} \| \\
&= (1+\delta_n)\| x_n - x^* \| + b_n
\end{aligned}
$$

where

$$\delta_n = \lambda_n^{(N)} + \lambda_n^{(N-1)}(1+\lambda_n^{(N)}) + \lambda_n^{(N-2)}(1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)}) + \ldots + \lambda_n^{(1)}(1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)})\ldots(1+\lambda_n^{(2)})$$

and

$$\begin{aligned} b_n = {} & (1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)})\ldots(1+\lambda_n^{(2)})\|v_n^{(1)}\| \\ & + (1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)})\ldots(1+\lambda_n^{(3)})\|v_n^{(2)}\| + \ldots \\ & + (1+\lambda_n^{(N)})(1+\lambda_n^{(N-1)})\|v_n^{(N-2)}\| + (1+\lambda_n^{(N)})\|v_n^{(N-1)}\| + \|v_n^{(N)}\|. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. It follows from Lemma 4 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. The proof is complete.

■

Lemma 9. Let E be a real uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with a retaction P. Let $T_1, T_2, ..., T_N : K \to E$ be asymptotically nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \phi$ and $\left\{k_n^{(i)}\right\}_{n=1}^{\infty}$ satisfy $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ for all $i \in \{1, 2, \ldots, N\}$. Let $\left\{\alpha_n^{(i)}\right\}_{n=1}^{\infty}$ be a sequence in $[\varepsilon, 1-\varepsilon], \varepsilon \in (0,1)$ for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence defined iteratively by (2). Then $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, N$.

Proof. By Lemma 8, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Let

$$\lim_{n\to\infty}\|x_n - x^*\| = c. \tag{3}$$

For any positive integer h with $2 \le h \le N$, we note that

$$\begin{aligned} \| x_n^{(h)} - x^* \| \le {} & \{1 + \lambda_n^{(h)} + \lambda_n^{(h-1)}(1+\lambda_n^{(h)}) + \lambda_n^{(h-2)}(1+\lambda_n^{(h)})(1+\lambda_n^{(h-1)}) + \ldots \\ & + \lambda_n^{(1)}(1+\lambda_n^{(h)})(1+\lambda_n^{(h-1)})\ldots(1+\lambda_n^{(2)})\}\|x_n - x^*\| + \| v_n^{(h)} \| \\ & + (1+\lambda_n^{(h)})\| v_n^{(h-1)} \| + (1+\lambda_n^{(h)} + \lambda_n^{(h-1)} + \lambda_n^{(h)}\lambda_n^{(h-1)})\| v_n^{(h-2)} \| + \ldots \\ & + \{1 + \lambda_n^{(h)} + \ldots + \lambda_n^{(2)} + \lambda_n^{(h)}\lambda_n^{(h-1)} + \ldots + \lambda_n^{(h)}\lambda_n^{(2)} \\ & + \lambda_n^{(h-1)}\lambda_n^{(h-2)} + \ldots + \lambda_n^{(h-1)}\lambda_n^{(2)}\}\| v_n^{(1)} \| \end{aligned} \tag{4}$$

and

$$\| T_h (PT_h)^{n-1} x_n^{(h-1)} - x^* \| \le (1+\lambda_n^{(h)})\| x_n^{(h-1)} - x^* \|. \tag{5}$$

From (4) and (5), we have

$$\begin{aligned}\limsup_{n\to\infty} \| x_n^{(h)} - x^* \| &\leq \limsup_{n\to\infty} \{[1+\lambda_n^{(h)}+\lambda_n^{(h-1)}(1+\lambda_n^{(h)}) \\ &\quad + \lambda_n^{(h-2)}(1+\lambda_n^{(h)})(1+\lambda_n^{(h-1)}) + \ldots \\ &\quad + \lambda_n^{(1)}(1+\lambda_n^{(h)})(1+\lambda_n^{(h-1)})\ldots(1+\lambda_n^{(2)})] \| x_n - x^* \| \\ &\quad + \| v_n^{(h)} \| + (1+\lambda_n^{(h)}) \| v_n^{(h-1)} \| + (1+\lambda_n^{(h)}+\lambda_n^{(h-1)} \\ &\quad + \lambda_n^{(h)}\lambda_n^{(h-1)}) \| v_n^{(h-2)} \| + \ldots + [1+\lambda_n^{(h)} + \ldots + \lambda_n^{(2)} \\ &\quad + \lambda_n^{(h)}\lambda_n^{(h-1)} + \ldots + \lambda_n^{(h)}\lambda_n^{(2)} + \lambda_n^{(h-1)}\lambda_n^{(h-2)} + \ldots \\ &\quad + \lambda_n^{(h-1)}\lambda_n^{(2)}] \| v_n^{(1)} \| \} \\ &\leq c. \end{aligned} \tag{6}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \| T_h(PT_h)^{n-1} x_n^{(h-1)} - x^* \| &\leq \limsup_{n\to\infty} \left\{ (1+\lambda_n^{(h)}) \| x_n^{(h-1)} - x^* \| \right\} \\ &\leq c. \end{aligned} \tag{7}$$

Moreover, we note that

$$\begin{aligned} c &= \lim_{n\to\infty} \| x_n - x^* \| \\ &= \lim_{n\to\infty} \| x_{n+1} - x^* \| \\ &\leq \lim_{n\to\infty} \| (1-\alpha_n^{(N)})(x_n - x^* + v_n^{(N)}) + \alpha_n^{(N)}(T_N(PT_N)^{n-1} x_n^{(N-1)} - x^* + v_n^{(N)}) \| \\ &\leq \lim_{n\to\infty}(1-\alpha_n^{(N)}) \| x_n - x^* \| + \lim_{n\to\infty}(1-\alpha_n^{(N)}) \| v_n^{(N)} \| \\ &\qquad + \limsup_{n\to\infty} \alpha_n^{(N)} \| T_N(PT_N)^{n-1} x_n^{(N-1)} - x^* \| + \lim_{n\to\infty} \alpha_n^{(N)} \| v_n^{(N)} \| \\ &= c. \end{aligned}$$

By Lemma 3, we obtain

$$\lim_{n\to\infty} \| x_n - T_N(PT_N)^{n-1} x_n^{(N-1)} \| = 0. \tag{8}$$

Hence, by (7), we also have

$$\limsup_{n\to\infty} \| T_N(PT_N)^{n-1} x_n^{(N-1)} - x^* \| \leq c.$$

Note that

$$\begin{aligned} \| x_n - x^* \| &\leq \| x_n - T_N(PT_N)^{n-1} x_n^{(N-1)} \| + \| T_N(PT_N)^{n-1} x_n^{(N-1)} - x^* \| \\ &\leq \| x_n - T_N(PT_N)^{n-1} x_n^{(N-1)} \| + (1+\lambda_n^{(N)}) \| x_n^{(N-1)} - x^* \|. \end{aligned}$$

Thus, by (8), we have

$$c = \liminf_{n\to\infty} \| x_n - x^* \| \leq \liminf_{n\to\infty} \| x_n^{(N-1)} - x^* \| \tag{9}$$

and hence

$$c \leq \liminf_{n\to\infty} \| x_n^{(N-1)} - x^* \| \leq \limsup_{n\to\infty} \| x_n^{(N-1)} - x^* \| \leq c.$$

This implies

$$\lim_{n\to\infty} \| x_n^{(N-1)} - x^* \| \;=\; c.$$

Hence

$$\begin{aligned} c \;=\; \lim_{n\to\infty} \| & (1-\alpha_n^{(N-1)})(x_n - x^* + v_n^{(N-1)}) \\ & + \alpha_n^{(N-1)}(T_{N-1}(PT_{N-1})^{n-1} x_n^{(N-2)} - x^* + v_n^{(N-1)}) \|. \end{aligned}$$

Similarly by (7) and Lemma 3, we obtain

$$\lim_{n\to\infty} \| x_n - T_{N-1}(PT_{N-1})^{n-1} x_n^{(N-2)} \| \;=\; 0. \tag{10}$$

Continuting in this fashion, we note that

$$\lim_{n\to\infty} \| x_n - T_h(PT_h)^{n-1} x_n^{(h-1)} \| \;=\; 0. \tag{11}$$

For all $h \in \{1, 2, \dots, N\}$, and so

$$\lim_{n\to\infty} \| x_n^{(h)} - x^* \| \;=\; c.$$

Note that

$$\begin{aligned} \| x_n - T_h(PT_h)^{n-1} x_n \| \;\le\; & \| x_n - T_h(PT_h)^{n-1} x_n^{(h-1)} \| \\ & + \| T_h(PT_h)^{n-1} x_n^{(h-1)} - T_h(PT_h)^{n-1} x_n \| \\ \le\; & \| x_n - T_h(PT_h)^{n-1} x_n^{(h-1)} \| + (1+\lambda_n^{(h)}) \| (1-\alpha_n^{(h-1)}) x_n \\ & + \alpha_n^{(h-1)} T_{h-1}(PT_{h-1})^{n-1} x_n^{(h-2)} + v_n^{(h-1)} - x_n \| \\ \le\; & \| x_n - T_h(PT_h)^{n-1} x_n^{(h-1)} \| \\ & + \alpha_n^{(h-1)}(1+\lambda_n^{(h)}) \| x_n - T_{h-1}(PT_{h-1})^{n-1} x_n^{(h-2)} \| \\ & + (1+\lambda_n^{(h)}) \| v_n^{(h-1)} \|. \end{aligned} \tag{12}$$

Thus, we have

$$\lim_{n\to\infty} \| x_n - T_h(PT_h)^{n-1} x_n \| \;=\; 0. \tag{13}$$

Since for each i, T_i is asymptotically nonexpansive, there exists a Lipschitzain constant $L_i > 0$ such that

$$\begin{aligned} \| x_n - T_h x_n \| \;\le\; & \| x_n - T_h(PT_h)^{n-1} x_n \| + \| T_h(PT_h)^{n-1} x_n - T_h(PT_h)^{n-1} x_n^{(h-1)} \| \\ & + \| T_h(PT_h)^{n-1} x_n^{(h-1)} - T_h x_n \| \\ \le\; & \| x_n - T_h(PT_h)^{n-1} x_n \| + (1+\lambda_n^{(h)}) \| x_n - x_n^{(h-1)} \| \\ & + L \| T_h(PT_h)^{n-2} x_n^{(h-1)} - x_n \| \end{aligned}$$

where $L \;=\; \max_{1\le i\le N}\{L_i\}$. From (12) and (13), we have

$$\lim_{n\to\infty} \| x_n - T_h x_n \| \;=\; 0.$$

For $\lim_{n\to\infty} \| x_n - T_1 x_n \| \;=\; 0,$ it follows that

$$\begin{aligned}\| x_n^{(1)} - x^* \| &= \|P((1-\alpha_n^{(1)})x_n + \alpha_n^{(1)}T_1(PT_1)^{n-1}x_n + v_n^{(1)}) - x^* \| \\ &\le (1-\alpha_n^{(1)})\| x_n - x^* \| + \alpha_n^{(1)}(1+\lambda_n^{(1)})\| x_n - x^* \| + \| v_n^{(1)} \| \\ &= (1+\alpha_n^{(1)}\lambda_n^{(1)})\| x_n - x^* \| + \| v_n^{(1)} \|.\end{aligned}$$

This implies

$$\limsup_{n\to\infty} \| x_n^{(1)} - x^* \| \le c. \tag{14}$$

Moreover, we note that

$$\begin{aligned}\| x_n - x^* \| &\le \| x_n - T_{N-1}(PT_{N-1})^{n-1}x_n^{(1)} \| + \| T_{N-1}(PT_{N-1})^{n-1}x_n^{(1)} - x^* \| \\ &\le \| x_n - T_{N-1}(PT_{N-1})^{n-1}x_n^{(1)} \| + (1+\lambda_n^{(N-1)})\| x_n^{(1)} - x^* \|.\end{aligned}$$

Then, by (11), we get

$$c \le \liminf_{n\to\infty} \| x_n^{(1)} - x^* \|. \tag{15}$$

From (14) and (15), we have

$$\lim_{n\to\infty} \| x_n^{(1)} - x^* \| = c.$$

Moreover, we note that

$$\begin{aligned}c &= \lim_{n\to\infty} \| x_n^{(1)} - x^* \| \\ &\le \lim_{n\to\infty} \| (1-\alpha_n^{(1)})x_n + \alpha_n^{(1)}T_1(PT_1)^{n-1}x_n + v_n^{(1)} - x^* \| \\ &= \lim_{n\to\infty} \| (1-\alpha_n^{(1)})(x_n - x^* + v_n^{(1)}) + \alpha_n^{(1)}(T_1(PT_1)^{n-1}x_n - x^* + v_n^{(1)}) \| \\ &\le c.\end{aligned}$$

By Lemma 3, we have

$$\lim_{n\to\infty} \| x_n - T_1(PT_1)^{n-1}x_n \| = 0.$$

Note that

$$\begin{aligned}\| x_n - T_1x_n \| &\le \| x_n - T_1(PT_1)^{n-1}x_n \| + \| T_1(PT_1)^{n-1}x_n - T_1x_n \| \\ &\le \| x_n - T_1(PT_1)^{n-1}x_n \| + L\| T_1(PT_1)^{n-2}x_n - x_n \|.\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} \| x_n - T_1x_n \| = 0.$$

Since

$$\begin{aligned}\| x_n - T_Nx_n \| &\le \|x_n - T_{N-1}(PT_{N-1})^{n-1}x_n^{(N-2)} \| + \| T_{N-1}(PT_{N-1})^{n-1}x_n^{(N-2)} - T_Nx_n \| \\ &\le \|x_n - T_{N-1}(PT_{N-1})^{n-1}x_n^{(N-2)} \| + L\| T_{N-1}(PT_{N-1})^{n-2}x_n^{(N-2)} - x_n \|,\end{aligned}$$

it follows that

$$\lim_{n\to\infty} \| x_n - T_Nx_n \| = 0.$$

The proof is complete.

■

Theorem 10. Let E be a real uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with a retaction P. Let $T_1, T_2, ..., T_N : K \to E$ be asymptotically nonexpansive mappings with sequences $\left\{k_n^{(i)}\right\}_{n=1}^{\infty}$ and $\left\{\alpha_n^{(i)}\right\}_{n=1}^{\infty}$ as in Lemma 9. If one of $\left\{T_i\right\}_{i=1}^{N}$ is either completely continuous or semicompact, then $\{x_n\}$ defined by (2) converges strongly to common fixed point of $\left\{T_i\right\}_{i=1}^{N}$.

Proof. If one of $\left\{T_i\right\}_{i=1}^{N}$ is semicompact, say T_s, s $\in \{1, 2, ..., N\}$ from the fact that $\lim_{n\to\infty} \| x_n - T_s x_n \| = 0$ and $\{x_n\}$ is bounded, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ that converges strongly to some $x^* \in K$. By Lemma 4 guarantees that $(I-T)x^* = 0$, i.e. $T_s x = x$. From $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0$ and the continuity of T_i, $i = 1, 2, .., N$, and using $x_{n_j} \to x$ as $j \to \infty$ we obtain that $x^* \in \bigcap_{i=1}^{N} F(T_i)$. By Lemma 8, $\lim_{n\to\infty} \| x_n - x^* \|$ exists and $\{x_n\}$ converges strongly to x^*.

If, on the other hand, one of $\left\{T_i\right\}_{i=1}^{N}$ is completely continuous, say T_s, then $\{T_s x_n\}$ is bounded, there exists a subsequence $\{T_s x_{n_j}\}$ converging strongly to some $\tilde{x}$. By Lemma 9, $\lim_{j\to\infty} \| x_{n_j} - T_s x_{n_j} \| = 0$, and by the continuity of T_s, we have $\lim_{j\to\infty} \| x_{n_j} - \tilde{x} \| = 0$. Using Lemma 4 as above, $\tilde{x} \in \bigcap_{i=1}^{N} F(T_i)$. Thus, since $\lim_{n\to\infty} \| x_n - \tilde{x} \|$ exists by Lemma 8, we have $\lim_{n\to\infty} \| x_n - \tilde{x} \| = 0$ and so $\{x_n\}$ converges strongly to $\tilde{x}$. The proof is complete.

■

Corollary 11. (Wang, 2006) Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2 : K \to K$ be nonself asymptotically nonexpansive mappings with sequences $\{k_n^1\}, \{k_n^2\}, \{\alpha_n^1\}$, and $\{\alpha_n^2\}$ as in Lemma 9. If one of T_1, T_2 is either completely continuous or semicompact, then $\{x_n\}$ defined by

$$
\begin{aligned}
& x_1 \in K \\
& x_n^{(1)} = (1-\alpha_n^{(1)})x_n + \alpha_n^{(1)}(T_1)^n x_n + v_n^{(1)}, \\
& x_n^{(2)} = (1-\alpha_n^{(2)})x_n + \alpha_n^{(2)}(T_2)^n x_n^{(1)} + v_n^{(2)}
\end{aligned}
\tag{16}
$$

converges strongly to common fixed point of T_1 and T_2.

We now prove weak convergence theorems.

Theorem 12. Let E be a real uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with a retraction P. Let $T_1, T_2, ..., T_N : K \to E$ be nonself asymptotically nonexpansive mappings with sequences $\left\{k_n^{(i)}\right\}_{n=1}^{\infty}$ and $\left\{\alpha_n^{(i)}\right\}_{n=1}^{\infty}$ as in Lemma 9. If E satisfies Opial's condition or has a Frechet differentiable norm, then $\{x_n\}$ defined by (2) converges weakly to common fixed point of $\left\{T_i\right\}_{i=1}^{N}$.

Proof. If E satisfies Opial's condition the proof follows as in the proof of Theorem 3.2 of Takahashi and Tamura (1998). If E has Frechet differentiable norm, the proof of Theorem 3.10 of Chidume, Ofoedu, and Zegeye (2003), using Lemma 7 instead of Lemma 3.9 of Chidume et al. (2003).

■

Corollary 3.6. (Wang, 2006) Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2 : K \to K$ be nonself asymptotically nonexpansive mappings with sequences $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{\alpha_n^{(1)}\}$,and $\{\alpha_n^{(2)}\}$ as in Lemma 9. If E satisfies Opial's condition or has a Frechet differentiable norm, then $\{x_n\}$ defined by (16) converges weakly to common fixed point of of T_1 and T_2.

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