On the Stability of Some Pexiderized Fuzzy Number-Valued Functional Equations

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ABSTRACT

The main goal of this paper is to investigate the Ulam stability of the new pexider fuzzy number-valued functional equation via the metric related to the Hausdorff metric defined on the class of special alpha-cuts of fuzzy numbers, where unknown variables are fuzzy number-valued functions on a subspace of a Banach space. Our main result covers many cases of stability results of the fuzzy number-valued functional equations.

Keywords: *Functional equation, Ulam stability, Fuzzy number-valued mapping, Hausdorff metric*

INTRODUCTION AND PRELIMINARIES

The stability problem of a functional equation which is related to a question of Ulam (1940) concerning the group homomorphism was first partial answered by Hyers (1941) for Banach spaces. Nowadays, many researchers studied this result and investigated several stability results in many directions on various abstract spaces.

In 2011, the stability problem of the following functional equation was investigated by Eshaghi et al. (2011):

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f\left(x\right), \tag{1.1}$$

where f is a single value function. Recently, Wu and Jin (2019) first studied the connection between the Ulam stability and the fuzzy number-valued functional equation. They proved the stability of the above functional equation, where f is a fuzzy number-valued mapping on a Banach space.

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Next, we recall some necessary notions and fundamental results which are needed to prove the main result in this paper.

In this paper, $\mathbb{N}, \mathbb{R}, \mathbb{R}^{+}$ denote the set of all natural numbers, the set of all real numbers, and the set of all positive real numbers, respectively, X and Y represent Banach spaces, $P_{kc}(X)$ denotes the set of all non-empty compact convex subsets of X, B is a subspace of Y.

Definition 1.1. Let X be a Banach space and $u: X \to [0,1]$ be a function satisfying the following conditions:

(i) $[u]^{\alpha} = \{x \in X : u(x) \geq \alpha\} \in P_{kc}(X)$ for all $\alpha \in (0,1]$;

(ii) $[u]^{0} = cl\{x \in X : u(x) > 0\}$ is a compact set, where the notation "cl" denotes the closure.

Then u is called a **fuzzy number** on X. The set of all fuzzy numbers on X is denoted by X_F.

For $u, v \in X_F$, $\lambda \in \mathbb{R}$, the following properties concerning the addition $u+v$ and the scalar multiplication $\lambda \cdot u$ can be proven via the Zadeh extension principle:

$$[u+v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha} \text{ and } [\lambda \cdot u]^{\alpha} = \lambda [u]^{\alpha}.$$

If $u, v \in X_F$, we can define the distance between u and v by

$$D(u,v) = \sup_{\alpha \in [0,1]} d_H\left([u]^{\alpha}, [v]^{\alpha}\right),$$

where d_H is the Hausdorff metric. It is well known that (X_F, D) is a complete metric space, and D satisfies the following properties for all $\lambda \in \mathbb{R}$ and $u, v, w, e \in X_F$:

(P1) $D(\lambda u, \lambda v) = |\lambda| \cdot D(u,v)$;

(P2) $D(u+w, v+w) = D(u,v)$;

(P3) $D(u+v, w+e) \leq D(u,w) + D(v,e)$.

In this article, we introduce the pexider fuzzy number-valued functional equation which is a generalization of Equation (1.1) as follows:

$$rf\left(\frac{x+y}{s}\right)+sg\left(\frac{x-y}{r}\right)=\frac{2r}{s}h(x), \tag{1.2}$$

where f,g,h are unknown fuzzy number-valued functions on a subspace of a Banach space, r and s are given non-zero real numbers. The Ulam stability results for the proposed fuzzy number-valued functional equation are proved.

MAIN RESULTS

In this section, we prove the stability of the functional equation (1.2) which is a generalization of one of main results in (Wu & Jin, 2019).

Theorem 2.1. If fuzzy number-valued mappings $f,g,h:B\to X_F$ satisfy the inequality

$$D\left(rf\left(\frac{x+y}{s}\right)+sg\left(\frac{x-y}{r}\right),\frac{2r}{s}h(x)\right)<\varepsilon \tag{2.1}$$

for all $x,y\in B,$ where $\varepsilon>0,$ $r,s\in\mathbb{R}\setminus\{0\},$ then there exists a unique additive mapping $T:B\to X_F$ such that $D\big(T(x),h(x)\big)\le\frac{3\varepsilon+\delta}{2}\left|\frac{s}{r}\right|$ for all $x\in B,$ where $\delta:=D\big(rf(0),-sg(0)\big).$

Moreover, if $h(tx)$ is continuous at $t\in\mathbb{R}$ for each given $x\in B,$ then T is linear on $B.$ Meanwhile, we obtain

$$D\left(f(x)+\frac{s}{r}g(0),T(x)\right)<\frac{4\varepsilon+\delta}{|r|}$$

and

$$D\left(g(x)+\frac{r}{s}f(0),\frac{r^2}{s^2}T(x)\right)<\frac{4\varepsilon+\delta}{|s|}$$

for all $x\in B.$

Proof. Letting $y=0,$ $y=x$ and $y=-x,$ respectively, in (2.1), the following inequalities hold:

$$D\left(rf\left(\frac{x}{s}\right)+sg\left(\frac{x}{r}\right),\frac{2r}{s}h(x)\right)<\varepsilon, \tag{2.2}$$

$$D\left(rf\left(\frac{2x}{s}\right)+sg(0),\frac{2r}{s}h(x)\right)<\varepsilon, \tag{2.3}$$

$$D\left(rf(0)+sg\left(\frac{2x}{r}\right),\frac{2r}{s}h(x)\right)<\varepsilon \tag{2.4}$$

for all $x\in B.$ It follows from (P1)-(P3) together with (2.2)-(2.4) that

$$\begin{aligned}
&D\left(h(x),\frac{1}{2}h(2x)\right)\\
&=\frac{1}{4\left|\frac{r}{s}\right|}D\left(\frac{4r}{s}h(x),\frac{2r}{s}h(2x)\right)\\
&\le\frac{1}{4\left|\frac{r}{s}\right|}D\left(rf\left(\frac{2x}{s}\right)+sg\left(\frac{2x}{r}\right),\frac{2r}{s}h(2x)\right)\\
&\quad+\frac{1}{4\left|\frac{r}{s}\right|}D\left(\frac{2r}{s}h(x),rf\left(\frac{2x}{s}\right)+sg(0)\right)\\
&\quad+\frac{1}{4\left|\frac{r}{s}\right|}D\left(\frac{2r}{s}h(x),sg\left(\frac{2x}{r}\right)+rf(0)\right)\\
&\quad+\frac{1}{4\left|\frac{r}{s}\right|}D\big(rf(0),-sg(0)\big)\\
&<\frac{3\varepsilon+\delta}{4\left|\frac{r}{s}\right|}
\end{aligned}$$

for all $x\in B.$ Setting $h_0(x)=h(x)$ and $h_n(x)=\frac{1}{2^n}h(2^n x)$ for all $n\in\mathbb{N},$ we get the following inequality:

$$D\left(h_n(x),h_{n-1}(x)\right)=\frac{1}{2^{n-1}}D\left(\frac{1}{2}h\left(2^n x\right),h\left(2^{n-1}x\right)\right)<\frac{3\varepsilon+\delta}{2^{n+1}\left|\frac{r}{s}\right|} \tag{2.5}$$

for all $x\in B$ and for all $n\in\mathbb{N}$. By using (2.5), we can show that $D\left(h_n(x),h_m(x)\right)\to 0$ as $m,n\to\infty$. It yields that $\{h_n(x)\}$ is a Cauchy sequence in X_F for every $x\in B$. By the completeness of (X_F,D), we can construct a mapping $T:B\to X_F$ by $T(x)=\lim_{n\to\infty}h_n(x)$ for each $x\in B$.

Next, we will show that T is additive. From Equations (2.1)-(2.4), we obtain

$$\begin{aligned}
&D\left(h_n\left(\frac{x+y}{2}\right)+h_n\left(\frac{x-y}{2}\right),h_n(x)\right)\\
&=D\left(\frac{1}{2^n}h\left(\frac{2^n x+2^n y}{2}\right)+\frac{1}{2^n}h\left(\frac{2^n x-2^n y}{2}\right),\frac{1}{2^n}h\left(2^n x\right)\right)\\
&=\frac{1}{2^n}D\left(h\left(\frac{2^n x+2^n y}{2}\right)+h\left(\frac{2^n x-2^n y}{2}\right),h\left(2^n x\right)\right)\\
&=\frac{1}{2^n\left|\frac{2r}{s}\right|}D\left(\frac{2r}{s}h\left(\frac{2^n x+2^n y}{2}\right)+\frac{2r}{s}h\left(\frac{2^n x-2^n y}{2}\right),\frac{2r}{s}h\left(2^n x\right)\right)\\
&\le\frac{1}{2^n\left|\frac{2r}{s}\right|}D\left(\frac{2r}{s}h\left(\frac{2^n x+2^n y}{2}\right),rf\left(\frac{2^n x+2^n y}{s}\right)+sg(0)\right)\\
&\quad+\frac{1}{2^n\left|\frac{2r}{s}\right|}D\left(\frac{2r}{s}h\left(\frac{2^n x-2^n y}{2}\right),sg\left(\frac{2^n x-2^n y}{r}\right)+rf(0)\right)\\
&\quad+\frac{1}{2^n\left|\frac{2r}{s}\right|}D\left(rf\left(\frac{2^n x+2^n y}{s}\right)+sg\left(\frac{2^n x-2^n y}{r}\right),\frac{2r}{s}h\left(2^n x\right)\right)\\
&\quad+\frac{1}{2^n\left|\frac{2r}{s}\right|}D\left(rf(0),-sg(0)\right)\\
&<\frac{3\varepsilon+\delta}{2^n\left|\frac{2r}{s}\right|}
\end{aligned}$$

$$=\frac{3\varepsilon+\delta}{2^{n+1}\left|\frac{r}{s}\right|} \tag{2.6}$$

for all $x, y \in B$ and for all $n \in \mathbb{N}$. By taking the limit in the above inequality, we obtain

$$D\left(T\left(\frac{x+y}{2}\right)+T\left(\frac{x-y}{2}\right),T(x)\right)=\lim_{n\to\infty}D\left(h_n\left(\frac{x+y}{2}\right)+h_n\left(\frac{x-y}{2}\right),h_n(x)\right)=0$$

for all $x, y \in B$. It follows that $T(x+y)=T(x)+T(y)$ for all $x, y \in B$, that is, T is additive. Based on the inequality (2.5), we have

$$\begin{aligned}D\left(h(x),T(x)\right)&=\lim_{n\to\infty}D\left(h(x),h_n(x)\right)\\&\le\lim_{n\to\infty}\sum_{i=1}^{n}D\left(h_{i-1}(x),h_i(x)\right)\\&\le\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3\varepsilon+\delta}{2^{i+1}\left|\frac{r}{s}\right|}\\&=\frac{3\varepsilon+\delta}{2}\left|\frac{s}{r}\right|\end{aligned}$$

for all $x \in B$.

In addition, if $h(tx)$ is continuous at $t \in \mathbb{R}$ for each given $x \in B$, then

$$\begin{aligned}\lim_{a\to a_0}T(ax)&=\lim_{a\to a_0}\lim_{n\to\infty}\frac{1}{2^n}h\left(2^n ax\right)\\&=\lim_{n\to\infty}\lim_{a\to a_0}\frac{1}{2^n}h\left(2^n ax\right)\\&=\lim_{n\to\infty}\frac{1}{2^n}h\left(2^n a_0 x\right)\\&=T(a_0 x)\end{aligned} \tag{2.8}$$

for each $a_0 \in \mathbb{R}$ and $x \in B$. By the additiveness, $T(cx)=cT(x)$ for each rational number $c \in \mathbb{R}$ and $x \in B$. This fact, together with (2.8), ensures that $T(cx)=cT(x)$ for each $c \in \mathbb{R}$ and $x \in B$. As a result, T is linear on B.

Next, from the linearity of T and the inequality (2.3), we get

$$\begin{aligned}
&D\left(f(x)+\frac{s}{r}g(0),T(x)\right)\\
&=D\left(f(x)+\frac{s}{r}g(0)+\frac{2}{s}h\left(\frac{sx}{2}\right)+\frac{2}{s}T\left(\frac{sx}{2}\right),T(x)+\frac{2}{s}h\left(\frac{sx}{2}\right)+\frac{2}{s}T\left(\frac{sx}{2}\right)\right)\\
&\le D\left(f(x)+\frac{s}{r}g(0),\frac{2}{s}h\left(\frac{sx}{2}\right)\right)+D\left(\frac{2}{s}h\left(\frac{sx}{2}\right),\frac{2}{s}T\left(\frac{sx}{2}\right)\right)+D\left(\frac{2}{s}T\left(\frac{sx}{2}\right),T(x)\right)\\
&=\frac{1}{|r|}D\left(rf(x)+sg(0),\frac{2r}{s}h\left(\frac{sx}{2}\right)\right)+\frac{2}{|s|}D\left(h\left(\frac{sx}{2}\right),T\left(\frac{sx}{2}\right)\right)+D\left(T(x),T(x)\right)\\
&<\frac{\varepsilon}{|r|}+\frac{3\varepsilon+\delta}{|r|}\\
&=\frac{4\varepsilon+\delta}{|r|}
\end{aligned}$$

for all $x\in B.$ Similarly, from the linearity of T and the inequality (2.4), we get

$$\begin{aligned}
&D\left(g(x)+\frac{r}{s}f(0),\frac{r^2}{s^2}T(x)\right)\\
&=D\left(g(x)+\frac{r}{s}f(0)+\frac{2r}{s^2}h\left(\frac{rx}{2}\right)+\frac{2r}{s^2}T\left(\frac{rx}{2}\right),\frac{r^2}{s^2}T(x)+\frac{2r}{s^2}h\left(\frac{rx}{2}\right)+\frac{2r}{s^2}T\left(\frac{rx}{2}\right)\right)\\
&\le D\left(g(x)+\frac{r}{s}f(0),\frac{2r}{s^2}h\left(\frac{rx}{2}\right)\right)+D\left(\frac{2r}{s^2}h\left(\frac{rx}{2}\right),\frac{2r}{s^2}T\left(\frac{rx}{2}\right)\right)\\
&\quad+D\left(\frac{2r}{s^2}T\left(\frac{rx}{2}\right),\frac{r^2}{s^2}T(x)\right)\\
&=\frac{1}{|s|}D\left(sg(x)+rf(0),\frac{2r}{s}h\left(\frac{rx}{2}\right)\right)+\frac{2|r|}{s^2}D\left(h\left(\frac{rx}{2}\right),T\left(\frac{rx}{2}\right)\right)\\
&\quad+D\left(\frac{r^2}{s^2}T(x),\frac{r^2}{s^2}T(x)\right)\\
&<\frac{\varepsilon}{|s|}+\frac{3\varepsilon+\delta}{|s|}\\
&=\frac{4\varepsilon+\delta}{|s|}
\end{aligned}$$

for all $x\in B.$

Finally, we will prove that T is unique. Suppose that there are two additive mappings $T_1, T_2 : B \to X_F$ satisfying $D(h(x), T_i(x)) \le \frac{3\varepsilon + \delta}{2}\left|\frac{s}{r}\right|$ for all $x \in B$ and for all $i = 1, 2.$ For each $x \in B,$ we get

$$
\begin{aligned}
D(T_1(x), T_2(x)) &= \frac{1}{n} D(nT_1(x), nT_2(x)) \\
&\le \frac{1}{n}\left(D(h(nx), T_1(nx)) + D(h(nx), T_2(nx))\right) \\
&\le \frac{3\varepsilon + \delta}{n}\left|\frac{s}{r}\right|.
\end{aligned}
$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain $T_1(x) = T_2(x)$ for all $x \in B.$ This completes the proof. □

We end this section by the following results which are obtained from the above theorem.

Corollary 2.2. If fuzzy number-valued mappings $f, g, h : B \to X_F$ satisfy the inequality

$$
D\left(sf\left(\frac{x+y}{s}\right) + sg\left(\frac{x-y}{s}\right), 2h(x)\right) < \varepsilon
$$

for all $x, y \in B,$ where $\varepsilon > 0,$ $s \in \mathbb{R} \setminus \{0\},$ then there exists a unique additive mapping $T : B \to X_F$ such that $D(T(x), h(x)) \le \frac{3\varepsilon + \delta}{2}$ for all $x \in B,$ where $\delta := D(sf(0), -sg(0)).$

Moreover, if $h(tx)$ is continuous at $t \in \mathbb{R}$ for each given $x \in B,$ then T is linear on $B.$ Meanwhile, we obtain

$$
D(f(x) + g(0), T(x)) < \frac{4\varepsilon + \delta}{|s|}
$$

and

$$
D(g(x) + f(0), T(x)) < \frac{4\varepsilon + \delta}{|s|}
$$

for all $x \in B.$

Proof. By taking $r = s$ in Theorem 2.1, we get this result. □

Corollary 2.3. If fuzzy number-valued mappings $f, g, h : B \to X_F$ satisfy the inequality

$$D\left(f\left(\frac{x+y}{2}\right) + g\left(\frac{x-y}{2}\right), h(x)\right) < \varepsilon$$

for all $x, y \in B,$ where $\varepsilon > 0,$ then there exists a unique additive mapping $T : B \to X_F$ such that $D(T(x), h(x)) \le \frac{6\varepsilon + \delta}{2}$ for all $x \in B,$ where $\delta := D(2f(0), -2g(0)).$

Moreover, if $h(tx)$ is continuous at $t \in \mathbb{R}$ for each given $x \in B,$ then T is linear on $B.$ Meanwhile, we obtain

$$D(T(x), f(x) + g(0)) < \frac{8\varepsilon + \delta}{2}$$

and

$$D(g(x) + f(0), T(x)) < \frac{8\varepsilon + \delta}{2}$$

for all $x \in B.$

Proof. By taking $r = s = 2$ in Theorem 2.1, we get this result. □

If we take all unknown functions in the above corollary are same, we get the following result.

Corollary 2.4. If a fuzzy number-valued mapping $f : B \to X_F$ satisfies the inequality

$$D\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right), f(x)\right) < \varepsilon \tag{2.9}$$

for all $x, y \in B,$ where $\varepsilon > 0,$ then there exists a unique additive mapping $T : B \to X_F$ such that

$$D(T(x), f(x)) \le 5\varepsilon \tag{2.10}$$

for all $x \in B.$

Moreover, if $h(tx)$ is continuous at $t \in \mathbb{R}$ for each given $x \in B,$ then T is linear on $B.$

Proof. From (2.9), we get $D\left(2f(0), f(0)\right) < \varepsilon.$ This implies that $D\left(8f(0), 4f(0)\right) < 4\varepsilon$ and so $\delta := D\left(2f(0), -2f(0)\right) < 4\varepsilon.$ By using Corollary 2.3, we get this result. □

It points out that the right-hand side of (2.10) is obtained from Corollary 2.3. However, we can omit some step in the proof of Theorem 2.1 whenever $f = g = h.$ This leads to the following results which is the main result in (Wu & Jin, 2019).

Corollary 2.5 (Wu & Jin, 2019**).** If fuzzy number-valued mappings $f : B \to X_F$ satisfy the inequality

$$D\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right), f(x)\right) < \varepsilon$$

for all $x, y \in B,$ where $\varepsilon > 0,$ then there exists a unique additive mapping $T : B \to X_F$ such that $D\left(T(x), f(x)\right) \le \varepsilon$ for all $x \in B.$

Moreover, if $h(tx)$ is continuous at $t \in \mathbb{R}$ for each given $x \in B,$ then T is linear on $B.$

CONCLUSIONS

We established Ulam stability results of the following functional equation via the metric related to the Hausdorff metric defined on the class of special alpha-cuts of fuzzy numbers:

$$rf\left(\frac{x+y}{s}\right) + sg\left(\frac{x-y}{r}\right) = \frac{2r}{s} h(x),$$

where f, g, h are unknown fuzzy number-valued functions on a subspace of a Banach space, r and s are given non-zero real numbers. Furthermore, we have shown Ulam stability results from the reduced above functional equation.

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